3. Option pricing models

Using properties derived from simple arbitrage arguments allowed us to derive upper and lower bounds for European and American option prices. While these results improve our understanding of options they are not accurate enough to give us an exact pricing relationship. This is what we are going to explore now.

3.1 B&S option pricing formula

The first and most widely used formula for pricing options is the Black, Scholes and Merton formula, which gives an analytical expression for the exact value of **European** call and put options on a single stock. The model and associated call and put option formulas have revolutionized finance theory and practice, and the surviving inventors Merton and Scholes received the Nobel Prize in Economics in 1997 for their contributions.

The derivation of the formula is relatively complex, and we will not explore it here. It relies on several assumptions:

- Perfect continuous time markets: one can trade at any time, there are no transaction costs and taxes, no restrictions on short-selling;
- No arbitrage opportunities exist;
- The risk free interest rate is constant during the option's life
- The price volatility of the underlying asset is constant
- The price movements of the underlying instrument follow a lognormal distribution, which implies a normal distribution of the continuously compounded returns.
- The underlying asset does not pay any dividend or cash flow during the option's life.

The final formula is as follows for a European call option:

$$C = S \cdot N(d_1) - K \cdot e^{-r \cdot \tau} \cdot N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K \cdot e^{-r \cdot \tau}}\right)}{\sigma \cdot \sqrt{\tau}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{\tau} \text{ and } d_2 = d_1 - \sigma \cdot \sqrt{\tau}$$

where:

S	current spot price
τ	time to maturity of the call
K	strike price of the call
σ	stock's volatility
r	annual continuously compounded risk-free interest rate
$N(x)$	cumulative probability distribution function for a standardised normal variable (i.e. the probability that such a variable will be less than x, that is, the area under the normal curve up to x). The value of N(x) for a given x can be found in Table 3-1.

Note that, with some algebra, d_1 can be rewritten as: $d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$.

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3-1: Standard normal distribution

Using the put-call parity, we can easily derive the formula for a European put option:

$$P = K \cdot e^{-r\cdot\tau} \cdot N(-d_2) - S \cdot N(-d_1)$$

Note that since the normal distribution is symmetrical about the mean and the surface below the bell-curve is equal to one, we can always use a table which is only defined for positive x-values and use the transformation: N(–x) = 1 – N(x) to obtain values for negative x's.

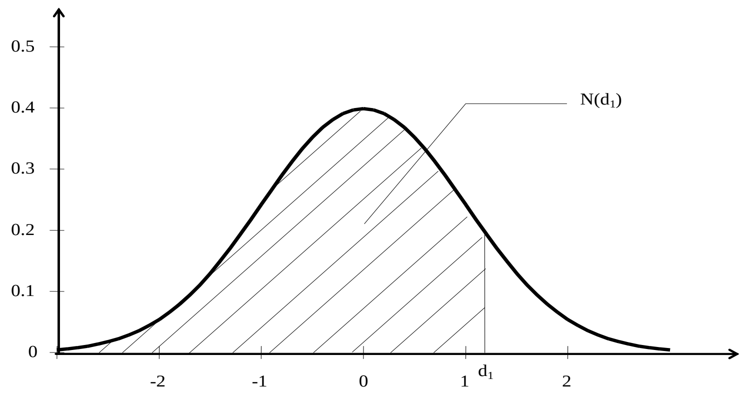

Figure 3-1: Interpreting the cumulative probability distribution function

Example:
What is the Black-Scholes price of a call option on a stock with the following characteristics?

- Stock price: S = CHF 280.
- Exercise price: K = CHF 260.
- Interest rate: R = 0.3% p.a. (simple interest rate)
- Time to maturity: 0.247 years [=90/365]
- Standard deviation of returns: 30% p.a.

First, we have to calculate the continuous compounded rate of return: $r = \ln(1 + R) \cong 0.3\%$, from which we can calculate the discounted strike price $K \cdot e^{-r \cdot \tau} = 260 \cdot e^{-0.003 \cdot 0.247} = 259.8$. Then, we can calculate the two parameters d_1 and d_2:

$$d_1 = \frac{\ln(\frac{S}{K \cdot e^{-r\cdot\tau}})}{\sigma \cdot \sqrt{\tau}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{\tau} = \frac{\ln(\frac{280}{259.8})}{0.3 \cdot \sqrt{0.247}} + \frac{1}{2} \cdot 0.3 \cdot \sqrt{0.247} = 0.576915$$

$$d_2 = d_1 - \sigma \cdot \sqrt{\tau} = 0.576915 - 0.3 \cdot \sqrt{0.247} = 0.427946$$

Next, we can find the values of $N(d_1)$ and $N(d_2)$ using a table:

$$\begin{aligned} N(d_1) &= N(0.576915) = N(0.57) + 0.6915 \cdot (N(0.58) - N(0.57)) \\ &= 0.7157 + 0.6915 \cdot (0.7190 - 0.7157) = 0.7180 \\ N(d_2) &= N(0.427946) = N(0.42) + 0.7946 \cdot (N(0.43) - N(0.42)) \\ &= 0.6628 + 0.7946 \cdot (0.6664 - 0.6628) = 0.6657. \end{aligned}$$

Thus, the Black-Scholes value of the call option is:

$$C = 280 \cdot 0.7180 - 259.8 \cdot 0.6657 = \text{CHF } 28.09$$

The intuitive explanation for the Black-Scholes formula is the following:

- the first term in the formula is the expected present value (using risk-neutral probabilities[6]) of the stock price conditional on the option finishing in-the-money: $S_T > K$. In that sense, the $N(d_i)$ terms are the risk adjusted probabilities that the call option will expire in the money;
- the second term in the formula is the present value of the expected exercise cost where the expectation is again taken with respect to risk-neutral probabilities.

It is interesting to note what the only variables required to value the option are: the volatility of the stock price, the risk-free rate of interest, the time to maturity, the strike price and the stock price. **The expected return on the underlying stock is not required** to calculate option values since only risk-neutral probabilities are used.

As we could expect from this interpretation, as S gets very large relative to K, then d_1, $d_2 \rightarrow \infty$ and $N(d_1)$, $N(d_2) \rightarrow 1$ such that $C \approx S - K \cdot e^{-r \cdot \tau}$.

3.2 European options on stocks paying known dividends

Let us now assume that the underlying stock pays a known dividend during the option's life. This dividend will be paid to the stock holder, and the call holder will not receive any part of it. But when the stock goes ex-dividend, its value will normally decrease by approximately the amount of the dividend distribution. This means that although the call holder started with a call on the cum-dividend stock, he will end up with a call on the ex-dividend stock. To take this dividend effect into account, the simplest approach is to replace the stock price S_0 in the Black-Scholes formula by the stock price minus the present value of the expected dividend[7].

In a more general case, if we assume that there are a total number of I dividends paid during the option's life, the new stock price to be used in the Black-Scholes formula would be the original stock price minus the present value of all future expected dividends. That is,

$$C_E = (S - \sum_{i=1}^{I} D_i \cdot e^{-r \cdot \tau_i}) \cdot N(d_1^*) - K \cdot e^{-r \cdot \tau} \cdot N(d_2^*)$$

$$P_E = K \cdot e^{-r \cdot \tau} \cdot N(-d_2^*) - (S - \sum_{i=1}^{I} D_i \cdot e^{-r \cdot \tau_i}) \cdot N(-d_1^*)$$

$$d_1^* = \frac{\ln\left[\left(S - \sum_{i=1}^{I} D_i \cdot e^{-r \cdot \tau_i}\right) \Big/ K \cdot e^{-r \cdot \tau}\right]}{\sigma \cdot \sqrt{\tau}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{\tau} \text{ and } d_2^* = d_1^* - \sigma \cdot \sqrt{\tau}$$

6 The notion of risk-neutral probability will be explained hereafter.

7 Note that from a theoretical perspective, because the stock price is lowered, this approach will typically lead to too little absolute price volatility ($\sigma \cdot S_t$) in the period before the dividend is paid. Several academic papers have suggested fixes to this problem which are based on adjustments of the volatility.

with the usual notations and:

- t_i i^{th}-dividend payment date
- $\tau = T - t$ time to maturity of the option
- $\tau_i = t_i - t$ time remaining until the i^{th} dividend payment date
- D_i dividend paid at time t_i

Let us illustrate this by an example.

Example:
On the 8th of February 2000, consider the European call option on stock XYZ expiring on the 25th of March (that is, $T - t = 0.125683$). The stock price is CHF 51.7, the strike price CHF 52, the volatility of the stock's returns 0.1235. The stock will pay a dividend D_1 = CHF 1.50 on the 15th of March (that is, τ_1 equals 36 / 366 = 0.09836) and the risk-free rate is r = 5.61%.

We have $d_1^* = -0.617934$, $d_2^* = -0.661717$, which gives $N(d_1^*) = 0.268282$, and $N(d_2^*) = 0.254051$. Thus the call price is:

$$C = (51.7 - 1.5 \cdot e^{-0.0561 \cdot 0.09836}) \cdot 0.268282 - 52 \cdot e^{-0.0561 \cdot 0.125683} \cdot 0.254051 = \text{CHF } 0.35$$

while it would be CHF 0.93 without any dividend.

This formula can also be applied to American calls if there is no risk of early exercise.

3.3 European options on stocks paying unknown dividends

A similar argument enables results obtained for European options written on non-dividend paying stocks to be extended so that they apply to European options on instruments paying a known **constant dividend yield**. The key idea is that the payment of a continuous dividend at rate y causes the growth rate in the stock price to be less than it would be otherwise by an amount y. That is, holding a stock which pays a continuous dividend yield of y and grows from S_t at time t to S_T at time T is equivalent to holding a non-dividend paying stock which grows from S_t at time t to $S_T \cdot e^{+y \cdot (T-t)}$ at time T, or, alternatively, from $S_t \cdot e^{-y \cdot (T-t)}$ at time t to S_T at time T.

Thus, a European option on a stock with price S paying a continuous dividend yield *y* has the same value as the corresponding European option on a stock with price $S \cdot e^{-y \cdot (T-t)}$ that pays no dividend. We can therefore adjust the original Black-Scholes formula to consider a "deflated" stock price.

This gives:

$$C_E = S \cdot e^{-y \cdot \tau} \cdot N(d_1^{'}) - K \cdot e^{-r \cdot \tau} \cdot N(d_2^{'})$$

$$P_E = K \cdot e^{-r \cdot \tau} \cdot N(-d_2^{'}) - (S \cdot e^{-y \cdot \tau}) \cdot N(-d_1^{'})$$

$$d_1^{'} = \frac{\ln\left(\frac{S}{K}\right) + (r - y) \cdot \tau}{\sigma \cdot \sqrt{\tau}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{\tau} \text{ and } d_2^{'} = d_1^{'} - \sigma \cdot \sqrt{\tau}$$

with the usual notations and:

- y   continuous dividend yield

In practice, in the case of single stocks, the dividend yield approach is used when the precise dividends' amounts and/or payment dates are unknown; a common practice is then to assume a **constant dividend yield "y"**. A rough way to estimate it is:

$$y = \frac{\text{dividend}}{\text{average quoted price}}$$

As we will see later, the hypothesis of a constant dividend yield will enable the application of this pricing formula to options on stock indices, on futures, on currencies, etc.

3.4 American options on stocks paying known dividends

In the case of American options, various cases have to be considered:

- if the underlying stock pays no dividend, the American call price is the same as the equivalent European call price. Then, the traditional Black-Scholes pricing can be applied. This is particularly useful for most short-term American calls;
- if the underlying stock pays some dividend but the no-early-exercise condition holds (Property 13), the Black-Scholes formula adapted for dividends can be used[8];
- if the underlying stock pays some dividend and early exercise cannot be excluded, the Black-Scholes formula cannot be applied. However, Fisher Black (1975) has developed a methodology, called the "pseudo-American call", in which **the American call can be priced as the maximum price in a portfolio of European calls with different maturities corresponding to the time of dividend payments**. The result is an estimated call price that draws on the intuition of the Black-Scholes model.

For American put options, as we have seen, an early exercise cannot be excluded, even if the underlying asset does not pay any dividend and Black-Scholes cannot be used. Thus, the binomial pricing model[9] or more complex methodologies have to be applied.

3.5 Options on stock indices

Theoretically, a call option on a stock index would give its holder the right (but not the obligation) to purchase the stock index (the underlying asset) by a certain date for a specified price, while a put option on a stock index would give its holder the right (but not the obligation) to sell the index by a certain date for a specified price.

As a stock index is not physically deliverable, a cash settlement has to be established. Thus, the main difference between calls on stock indices and calls on stocks is the settlement procedure:

- the holder of an in-the-money call on a single stock could receive a given number of stocks (depending on the contract size) against the payment of the strike price, while the holder of an in-the-money call on a stock index will receive the difference between the index value and the strike price, multiplied by the contract size;

8 But if dividends are unknown and that we make the assumption of a constant dividend yield, an early exercise cannot be excluded.

9 Binomial pricing is developed hereafter.